Relations between asymptotic and Fredholm representations*

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Abstract

We prove that for matrix algebras M_n there exists a monomorphism

$$(\prod_n M_n / \oplus_n M_n) \otimes C(S^1) \longrightarrow \mathcal{Q}$$

into the Calkin algebra which induces an isomorphism of the K_1-groups. As a consequence we show that every vector bundle over a classifying space $B\pi$ which can be obtained from an asymptotic representation of a discrete group π can be obtained also from a representation of the group $\pi \times \mathbf{Z}$ into the Calkin algebra. We give also a generalization of the notion of Fredholm representation and show that asymptotic representations can be viewed as asymptotic Fredholm representations.

1 Asymptotic representations as representations into the Calkin algebra

Let π be a discrete finitely presented group, and let $F \subset \pi$ be a finite subset. Denote by $U(n)$ the unitary group of dimension n and fix a number $\varepsilon > 0$.

Definition 1.1 A map $\sigma : \pi \longrightarrow U(n)$ is called an *ε-almost representation* with respect to F if $\sigma(g^{-1}) = \sigma(g)^{-1}$ holds for all $g \in \pi$ and if

$$\|\sigma\|_F = \sup\{\|\sigma(gh) - \sigma(g)\sigma(h)\| : g, h, gh \in F\} \leq \varepsilon.$$

Let $\{n_k\}$ be a strictly increasing sequence of positive integers and let $\sigma = \{\sigma_k : \pi \longrightarrow U(n_k)\}$ be a sequence of ε_k-almost representations. We assume that the groups $U(n_k)$ are embedded into the groups $U(n_{k+1})$ in the standard way, so it makes possible to compare almost representations for different k. Then we can consider the maps $\sigma_k \oplus 1 : \pi \longrightarrow U(n_k) \oplus U(n_{k+1} - n_k) \longrightarrow U(n_{k+1})$, which we also denote by σ_k.

Definition 1.2 A sequence of ε_k-almost representations is called an *asymptotic representation* of the group π (with respect to the finite subset F and a sequence $\{n_k\}$) if the sequences ε_k and $\|\sigma_k(g) - \sigma_{k+1}(g)\| : g \in F \subset \pi$ tend to zero.

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It was shown in [7] that this definition is equivalent to the definition given in [3, 4]. It does not depend on the choice of the finite subset F when F is big enough and sets of generators and relations of the group π are finite.

Two asymptotic representations σ_0 and σ_1 are called homotopic if there exists a family of asymptotic representations $\sigma_t = \{\sigma_{t,k}\}$ such that the functions $\sigma_{t,k}(g)$ are continuous for all $g \in \pi$ and $\lim_{k\to\infty} \max_t \|\sigma_{t,k}\|_F = 0$. It can be easily seen that every asymptotic representation is homotopic to some asymptotic representation corresponding to a given sequence $\{n_k\}$.

It is well known that the asymptotic representations are exact representations in some more compound C^*-algebras [4, 5]. Remind this construction. Let M_n be the $n \times n$ matrix algebra. Consider the C^*-algebra $B = \prod_{k=1}^{\infty} M_{n_k}$ of norm-bounded sequences of matrices. We suppose that the sequence n_k is strictly increasing. Denote by B^+ the C^*-algebra B with adjoined unit. Both algebras B and B^+ contain a C^*-ideal $I = \oplus_{k=1}^{\infty} M_{n_k}$ of sequences of matrices with norms tending to zero.

Denote the corresponding quotient algebras by $Q = B/I$ and $Q^+ = B^+/I$. Let $\bar{\alpha} : B^{(+)} \longrightarrow B^{(+)}$ be the right shift, $\bar{\alpha}(m_1, m_2, \ldots) = (0, m_1, m_2, \ldots)$, $(m_i) \in B^{(+)}$. As $\bar{\alpha}(I) \subset I$, so the homomorphism $\bar{\alpha}$ induces the homomorphism $\alpha : Q^{(+)} \longrightarrow Q^{(+)}$. Let

$$Q_\alpha^{(+)} = \{q \in Q^{(+)} : \alpha(q) = q\} \subset Q^{(+)}$$

be the α-invariant C^*-subalgebra. The adjoined unit gives the splittable short exact sequence

$$Q_\alpha \longrightarrow Q_\alpha^+ \longrightarrow \mathbf{C}. \tag{1.1}$$

Let $e = (e_k) \in B$ be the sequence of diagonal matrices having unities at the first place and zeroes at the other places, $e_k \in M_{n_k}$.

Lemma 1.3 *The group $K_0(Q_\alpha^+)$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ with generators $[e]$ and $[1]$;*
$K_0(Q_\alpha) \cong \mathbf{Z}$ with generator $[e]$;
$K_1(Q_\alpha) = K_1(Q_\alpha^+) = 0$.

Proof. Let $p \in M_r(Q_\alpha)$ be a projection and let $p' \in M_r(B)$ be a projection which is the lift of p. Then $p' = (p_k)$ is a sequence of projections. This sequence is α-invariant, hence the rank of p_k is constant beginning from some k. Therefore the projection p is a multiple of the projection e for big enough k. The case of the group $K_1(Q_\alpha)$ can be considered by the similar way. •

Denote by $\mathcal{Q}$ the Calkin algebra.

Theorem 1.4 *There exists a monomorphism*

$$\psi : Q_\alpha^+ \otimes C(S^1) \longrightarrow \mathcal{Q} \tag{1.2}$$

such that the induced homomorphism

$$\psi_* : K_1(Q_\alpha \otimes C(S^1)) \longrightarrow K_1(\mathcal{Q}) \tag{1.3}$$

is an isomorphism.

Proof. We start with defining the homomorphism (1.2). Let V_k be a n_k-dimensional Hilbert space where the algebra M_{n_k} acts, with a fixed basis $\{e_k^{(i)}\}$, $i \leq n_k$. Fix the embeddings

$$j_k : V_{k-1} \longrightarrow V_k,$$

mapping the vectors $e_{k-1}^{(i)}$ into $e_k^{(i)}$. Denote by W_{k-1} the orthogonal complements,

$$V_k = j_k(V_{k-1}) \oplus W_{k-1}.$$

For $k = 1$ let $W_0 = V_1$. Put

$$H_k = V_k \oplus (\oplus_m W_{k,m}); \quad W_{k,m} \cong W_k,\ m \in \mathbf{N}; \quad H = \oplus H_k$$

and define a homomorphism ψ of the C^*-algebra Q_α^+ into the Calkin algebra $\mathcal{Q}$ of the Hilbert space H. Let $q \in Q_\alpha^+$ and let $q' = (q_k) \in B^+$ be a lift of q. Denote by $(q_k)_{ij}$ the matrix elements of the matrix q_k. By definition the limits $\lim_{i\to\infty}(q_k)_{ii}$ exist and are equal to each other for all k. Denote the value of these limits by $\lambda(q)$. Define $\psi(q)$ as an operator on H which acts on the spaces V_k by multiplication by the matrices q_k, and on the spaces $W_{k,m}$ by multiplication by $\lambda(q)$. Such operator is defined up to compact operators, hence it gives an element $\psi(q) \in \mathcal{Q}$. Consider now the C^*-algebra $Q_\alpha^+ \otimes C(S^1)$ of continuous Q_α^+-valued functions over a circle. The construction described above allows us to define a homomorphism $Q_\alpha^+ \otimes C(S^1) \longrightarrow \mathcal{Q}$. Put $\psi(q \otimes 1_{C(S^1)}) = \psi(q)$. It remains now to define the image of the function $u = e^{2\pi it}$ from $C(S^1)$. Define a Fredholm operator F (with zero index) on the Hilbert space H by the following conditions:

i) F maps every subspace V_k (except V_1) onto the subspace $V_{k-1} \oplus W_{k-1,1}$ so that $F(e_k^{(i)}) = e_{k-1}^{(i)}$ for $i < n_{k-1}$;

ii) F isomorphically maps the subspace V_1 onto $W_{0,1}$;

iii) F isomorphically maps every subspace $W_{k,m}$ onto the subspace $W_{k,m+1}$.

Thus defined operator F commutes with the image $\psi(Q_\alpha^+)$ modulo compacts, hence we get the needed homomorphism ψ if we put $\psi(1_{Q_\alpha^+} \otimes e^{2\pi it}) = F$.

Denote the basis vectors of the subspaces $W_{k,m}$ by $w_{k,m}^{(j)}$ with $n_{k-1}+1 \leq j \leq n_k$. Then we can represent the action of the operator F by the diagram

$$\begin{array}{cccccccccccccccc}
\dots & \leftarrow & w_{1,2}^{(1)} & \leftarrow & w_{1,1}^{(1)} & \leftarrow \;| & e_1^{(1)} & \leftarrow & e_2^{(1)} & \leftarrow & e_3^{(1)} & \leftarrow & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \;| & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \leftarrow & w_{1,2}^{(n_1)} & \leftarrow & w_{1,1}^{(n_1)} & \leftarrow \;| & e_1^{(n_1)} & \leftarrow & e_2^{(n_1)} & \leftarrow & e_3^{(n_1)} & \leftarrow & \dots \\
 & & & & & & \text{— — —} & & & & & & \\
\dots & \leftarrow & w_{2,3}^{(n_1+1)} & \leftarrow & w_{2,2}^{(n_1+1)} & \leftarrow & w_{2,1}^{(n_1+1)} & \leftarrow \;| & e_2^{(n_1+1)} & \leftarrow & e_3^{(n_1+1)} & \leftarrow & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \;| & \dots & \dots & \dots & \dots & \dots \\
\dots & \leftarrow & w_{2,3}^{(n_2)} & \leftarrow & w_{2,2}^{(n_2)} & \leftarrow & w_{2,1}^{(n_2)} & \leftarrow \;| & e_2^{(n_2)} & \leftarrow & e_3^{(n_2)} & \leftarrow & \dots \\
 & & & & & & & & \text{— — —} & & & & \\
\dots & \leftarrow & w_{3,4}^{(n_2+1)} & \leftarrow & w_{3,3}^{(n_2+1)} & \leftarrow & w_{3,2}^{(n_2+1)} & \leftarrow & w_{3,1}^{(n_2+1)} & \leftarrow \;| & e_3^{(n_2+1)} & \leftarrow & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \;| & \dots & \dots & \dots \\
\dots & \leftarrow & w_{3,4}^{(n_3)} & \leftarrow & w_{3,3}^{(n_3)} & \leftarrow & w_{3,2}^{(n_3)} & \leftarrow & w_{3,1}^{(n_3)} & \leftarrow \;| & e_3^{(n_3)} & \leftarrow & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}$$

It remains to show that the homomorphism (1.3) is an isomorphism. As $K_1(Q_\alpha^+) = 0$, so by the Künneth formula we have

$$K_1(Q_\alpha^+ \otimes C(S^1)) \cong K_0(Q_\alpha^+) \otimes K^1(S^1).$$

Let $[u] \in K^1(S^1)$ be a generator. Then the group $K_1(Q_\alpha^+ \otimes C(S^1))$ is generated by two elements: $[e] \otimes [u]$ and $[1] \otimes [u]$. As

$$\psi(1 \otimes u) = F \quad \text{and} \quad \operatorname{ind} F = 0,$$

so we obtain $\psi_*([1] \otimes [u]) = 0$. Compute $\psi_*([e] \otimes [u])$. By [2] one has

$$[e] \otimes [u] = [(1-e) \otimes 1 + e \otimes u],$$

hence

$$\begin{aligned} \psi_*([e] \otimes [u]) &= \psi_*([e] \otimes [u]) = \psi_*([(1-e) \otimes 1 + e \otimes u]) \\ &= [1 - \psi(e \otimes 1) + \psi(e \otimes u)] = [1 - \psi(e \otimes 1) + \psi(e \otimes 1)F]. \end{aligned}$$

The action of the operator

$$F' = 1 - \psi(e \otimes 1) + \psi(e \otimes 1)F$$

is given by the formula

$$F'(e_k^{(j)}) = \begin{cases} e_k^{(j)} & \text{for } \; j > 1, \\ e_{k-1}^{(j)} & \text{for } \; j = 1, k > 1, \\ 0 & \text{for } \; j = k = 1; \end{cases} \qquad F'(w_{k,m}^{(j)}) = w_{k,m}^{(j)},$$

so the image of F' coincides with the whole space H, and the kernel of F' is one-dimensional and is generated by the vector $e_1^{(1)}$. Therefore $\operatorname{ind} F' = 1$, hence the homomorphism ψ_* maps the generator of the group $K_1(Q_\alpha \otimes C(S^1)) \cong \mathbf{Z}$ into the generator of the group $K_1(\mathcal{Q})$. $\bullet$

Denote by $\mathcal{R}_a(\pi)$ the Grothendieck group of virtual asymptotic representations of the group π. Let $\widetilde{\mathcal{R}}_a(\pi)$ denote the kernel of the map

$$\mathcal{R}_a(\pi) \longrightarrow \mathcal{R}_a(e) \cong \mathbf{Z},$$

defined by the trivial representation (here e is the trivial group). Let $B\pi$ be the classifying space for the group π. Remember that a construction was defined in [7], which allows to obtain a vector bundle over $B\pi$ starting from an asymptotic representation. This construction gives a homomorphism

$$\phi : \widetilde{\mathcal{R}}_a(\pi) \longrightarrow K^0(B\pi) \tag{1.4}$$

which can be described as follows. Let $C^*[\pi]$ be the group C^*-algebra of the group π and let $\xi \in K^0_{C^*[\pi]}(B\pi)$ be the universal bundle. An asymptotic representation σ defines a homomorphism

$$\overline{\sigma} : C^*[\pi] \longrightarrow Q_\alpha^+,$$

which maps the universal bundle ξ into some element $\overline{\sigma}_*(\xi) \in K^0_{Q^+_\alpha}(B\pi)$. This homomorphism defines the homomorphism

$$\phi' : \mathcal{R}_a(\pi) \longrightarrow K^0_{Q^+_\alpha}(B\pi).$$

It follows from the lemma 1.3, from the Künneth formula and from (1.1) that the lower line of the diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{R}}_a(\pi) & \longrightarrow & \mathcal{R}_a(\pi) & \longrightarrow & \mathcal{R}_a(e) \\
\downarrow & & \downarrow \phi' & & \downarrow \phi'_e \\
K^0_{Q_\alpha}(B\pi) & \longrightarrow & K^0_{Q^+_\alpha}(B\pi) & \longrightarrow & K^0_{\mathbf{C}}(B\pi)
\end{array}$$

is exact, therefore the left vertical arrow is well-defined. As we have

$$K^0_{Q_\alpha}(B\pi) = K^0(B\pi) \otimes K_0(Q_\alpha) \cong K^0(B\pi),$$

so we can define ϕ to be this left vertical arrow after identifying $K^0_{Q_\alpha}(B\pi)$ with $K^0(B\pi)$. Notice that the image of the homomorphism ϕ'_e coincides with the subgroup in $K^0(B\pi)$ generated by the trivial representations.

As there exists a natural isomorphism

$$j : K^0_A(B\pi \times S^1 \times S^1) \xrightarrow{\sim} K^0_{A\otimes C(S^1)}(B\pi \times S^1)$$

for any C^*-algebra A, so multiplication by the Bott generator $\beta \in K^0(S^1 \times S^1)$ defines an inclusion

$$\begin{array}{rcccl}
\overline{\beta} : K^0_A(B\pi) & \xrightarrow{\otimes\beta} & K^0_A(B\pi \times S^1 \times S^1) & \xrightarrow{j} & K^0_{A\otimes C(S^1)}(B\pi \times S^1) \\
& & & = & K^0_{A\otimes C(S^1)}(B(\pi \times \mathbf{Z})).
\end{array} \tag{1.5}$$

In the case $A = Q_\alpha$ we will write $\overline{\beta}$ instead of $\overline{\beta}_{Q_\alpha}$.

Now denote by $\mathcal{R}_{\mathcal{Q}}(\pi)$ the group of (virtual) representations of the group π into the Calkin algebra. It is easily seen that the homomorphism (1.4) allows us to define a homomorphism

$$\overline{\psi} : \mathcal{R}_a(\pi) \longrightarrow \mathcal{R}_{\mathcal{Q}}(\pi \times \mathbf{Z}) \tag{1.6}$$

given by the formula $\overline{\psi}(\overline{\sigma}) = \psi(\overline{\sigma} \otimes id)$ for $\overline{\sigma} \in \mathcal{R}_a(\pi)$ and

$$\overline{\sigma} \otimes id : C^*[\pi \times \mathbf{Z}] \cong C^*[\pi] \otimes C(S^1) \longrightarrow Q^+_\alpha \otimes C(S^1).$$

Let $\eta \in K^0_{C^*[\pi\times\mathbf{Z}]}(B(\pi \times Z))$ be the universal bundle over $B(\pi \times \mathbf{Z}$. Then there exists also a homomorphism

$$f : \mathcal{R}_{\mathcal{Q}}(\pi \times \mathbf{Z}) \longrightarrow K^0_{\mathcal{Q}}(B(\pi \times \mathbf{Z})) \tag{1.7}$$

defined as the image of η over $B(\pi\times\mathbf{Z})$ under the representations into the Calkin algebra.

All these homomorphisms (1.3) – (1.6) can be represented by the diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{R}}_a(\pi) & \xrightarrow{\phi} & K^0_{Q_\alpha}(B\pi) & \xrightarrow{\overline{\beta}} & K^0_{Q_\alpha\otimes C(S^1)}(B\pi \times S^1) \\
\downarrow & & & & \downarrow \psi_* \\
\mathcal{R}_a(\pi) & \xrightarrow{\overline{\psi}} & \mathcal{R}_{\mathcal{Q}}(\pi \times \mathbf{Z}) & \xrightarrow{f} & K^0_{\mathcal{Q}}(B\pi \times S^1),
\end{array} \tag{1.8}$$

where the left vertical arrow is the inclusion.

Theorem 1.5 *The diagram (1.8) is commutative.*

Proof. Direct calculation. •

So now we have the homomorphism

$$\overline{\phi} : \widetilde{\mathcal{R}}_a(\pi) \longrightarrow K^0_{\mathcal{Q}}(B\pi \times S^1) \cong K^1(B\pi \times S^1). \tag{1.9}$$

Theorem 1.5 shows that every element of the group $K^0(B\pi)$ which can be obtained by an asymptotic representation of the fundamental group π can be obtained also by a representation of the group $\pi \times \mathbf{Z}$ into the Calkin algebra.

2 Asymptotic representations as Fredholm representations

Notice that the group $K^0_{\mathcal{Q}}(B\pi \times S^1)$ can be decomposed into direct sum:

$$K^0_{\mathcal{Q}}(B\pi \times S^1) = K^0_{\mathcal{Q}}(B\pi) \oplus K^0_{\mathcal{Q}}(B\pi \wedge S^1) \cong K^1(B\pi) \oplus K^0(B\pi) \tag{2.1}$$

induced by an inclusion map $i : s_0 \longrightarrow S^1$, where $s_0 \in S^1$, and the image of the homomorphism $\overline{\phi}$ (1.9) lies only in the second summand of (2.1). Indeed, consider the composition of the map $\overline{\phi}$ with the map $i^* : K^0_{\mathcal{Q}}(B\pi \times S^1) \longrightarrow K^0_{\mathcal{Q}}(B\pi)$. But as the multiplication by the Bott generator is involved in the map $\overline{\phi}$, so its composition with i^* gives the zero map. Therefore the image of the group $\widetilde{\mathcal{R}}_a(\pi)$ lies in $K^0(B\pi)$ and hence defines a (virtual) vector bundle over $B\pi$.

On the other hand the image of the map (1.7) need not be contained in the second summand of (2.1). It would be so if the representation of the group $\pi \times \mathbf{Z}$ into the Calkin algebra would be a part of a Fredholm representation of the group π [6]. Using the notion of asymptotic representations we can now give a generalization of the Fredholm representations which would also ensure that the image of such representations would lie in the second summand of (2.1).

Let $\rho : \pi \times \mathbf{Z} \longrightarrow \mathcal{Q}$ be a representation into the Calkin algebra and let $F \subset \pi$ be a finite subset of π containing its generators. Denote by $B(H)$ the algebra of bounded operators on a separable Hilbert space H. Let $q : B(H) \longrightarrow \mathcal{Q}$ be the canonical projection.

Definition 2.1 We call a map $\tau : \pi \longrightarrow B(H)$ an *ε-trivialization* for ρ if

i) $\|\tau(gh) - \tau(g)\tau(h)\| \leq \varepsilon$ for any $g, h \in F \subset \pi$,

ii) $q(\tau(g)) = \rho(g; 0)$ for any $g \in \pi$, $(g; 0) \in \pi \times \mathbf{Z}$.

Definition 2.2 Suppose that for every $\varepsilon > 0$ there exists an ε-trivialization τ_ε for ρ. Then the pair (τ_ε, ρ) is called an *asymptotic Fredholm representation*.

Let u be a generator of the group $\mathbf{Z}$. Notice that the image of the group π under ε-trivializations commutes with some Fredholm operator $F = \rho(0, u)$ modulo compacts. Denote the group of all asymptotic Fredholm representations by $\mathcal{R}_{aF}(\pi) \subset \mathcal{R}_{\mathcal{Q}}(\pi \times \mathbf{Z})$.

Proposition 2.3 *The image of $\mathcal{R}_{aF}$ under the map f (1.7) lies in the group $K^0(B\pi \wedge S^1)$.*

Proof. It was described in the paper [7] how to construct a bundle over $B\pi$ with the fibers isomorphic to the Hilbert space H and with the structural group $GL(H)$ starting from an almost representation τ_ε for small enough ε. To do so one should construct transition functions acting on the fibers ξ_x,

$$T_g(x) : \xi_x \longrightarrow \xi_{gx}$$

for $g \in \pi$, $x \in E\pi$. One should chose representatives $\{a_\alpha\}$ in each orbit of the set of vertices of $E\pi$ and define $T_g(a_\alpha) = \tau_\varepsilon(g)$. Take now an arbitrary vertex $b \in E\pi$. Then there exists such $h \in \pi$ that $b = h(a_\alpha)$ and we should put $T_g(b) = \tau_\varepsilon(gh)\tau_\varepsilon^{-1}(h)$. Further these transition functions should be extended by linearity to all simplexes of $E\pi$. But obviously $q(T_g(b)) = q(\tau_\varepsilon(gh)\tau_\varepsilon^{-1}(h)) = q(\tau_\varepsilon(g))$, hence after we pass to quotients, the transition functions $q(T_g(x))$ would become constant and the bundle with the structural group being the invertibles of the Calkin algebra. But as any bundle with fibers H is trivial, so the quotient bundle with fibers isomorphic to the Calkin algebra is trivial too and the projection of $\varphi(\mathcal{R}_{aF})$ onto the first summand of $K^0_{\mathcal{Q}}(B\pi)$ is equal to zero. •

Remark. In the section 1 we have seen that an asymptotic representation $\sigma = (\sigma_n)$ defines a homomorphism $\rho : \pi \times \mathbf{Z} \longrightarrow GL(\mathcal{Q})$ into the group of invertibles of the Calkin algebra. But the same asymptotic representation gives ε-trivializations of ρ for any $\varepsilon > 0$. Indeed we can put

$$\tau_{\varepsilon_n}(g) = (1, 1, \ldots, 1, \sigma_n(g), \sigma_{n+1}(g), \ldots).$$

So asymptotic representations define asymptotic Fredholm representations and we have an inclusion $\mathcal{R}_a(\pi) \subset \mathcal{R}_{aF}(\pi)$.

3 Asymptotic representations and extensions

It was shown in [4] that using a quasi central approximate unity one can construct an aymptotic representation out of an extension of C^*-algebras. We study how this construction is related to asymptotic representations of the initial C^*-algebra.

Let A be a C^*-algebra such that there exists a homomorphism $q_A : A \longrightarrow \mathbf{C}$ of A into the complex numbers and let $F \subset A$ be a finite set of generators for A. We can repeat our definitions of asymptotic representations from the section 1 in the case of C^*-algebras instead of discrete groups. For smplicity sake we assume that dimension of an almost representation σ_n equals to n.

We consider a sequence of maps $\widetilde{\sigma}_n : A \longrightarrow M_n$ where M_n acts on a finite-dimensional Hilbert space V_n. Take an infinite-dimensional Hilbert space $H_n \supset V_n$ and define a map $\sigma_n : A \longrightarrow B(H)$ by $\sigma_n(a) = \widetilde{\sigma}_n(a) \oplus q_A(a)$ where $a \in A$ and $q_A(a)$ is a scalar on $V_n^\perp$. The

map $\sigma = \oplus_n \sigma_n$ gives an asymptotic representation of the C^*-algebra A if (after identifying all H_n) the norms $\|\sigma_{n+1}(a) - \sigma_n(a)\|$ tend to zero for $a \in F$.

Consider the C^*-algebra $\mathcal{E}$ in the Hilbert space $\oplus_n H_n$ generated by $\sigma(a)$, $a \in A$ and by the translation operator F defined in the proof of the theorem 1.4. Then one has a short exact sequence

$$\oplus_n M_n \longrightarrow \mathcal{E} \longrightarrow A \otimes C(S^1). \tag{3.1}$$

We consider a discrete version of the construction of [4]. If $e_n \in \oplus_n M_n$ is a quasi central approximate unity [1] then the exact sequence (3.1) defines an asymptotic representation

$$\rho_n : A \otimes C(S^1) \otimes C_0(S^1) \xrightarrow{as} \oplus_n M_n \tag{3.2}$$

given by the formula

$$\rho_n(a \otimes g \otimes f) = (a \otimes g)' \cdot f(e_n),$$

where $(a \otimes g)' \in \mathcal{E}$ is a lift for $a \otimes g$. Denote by $K_n \subset \oplus_n H_n$ the subspace on which one has $e_n \neq 0$, $e_n \neq 1$. If K_n is finite-dimensional, $k(n) = \dim K_n$, then we can get a sequence of finite-dimensional almost representations

$$\overline{\rho}_n : A \otimes C(S^1) \otimes C_0(S^1) \xrightarrow{as} \oplus_n M_{k(n)}$$

by the formula

$$\overline{\rho}_n(a \otimes g \otimes f) = P_n(a \otimes g)' P_n \cdot f(e_n),$$

where P_n is the projection onto K_n, $g \in C(S^1)$, $f \in C_0(S^1)$, $f(0) = 0$.

There exists also a well-known asymptotic representation $\beta = (\beta_m)$ of the commutative C^*-algebra $C(S^1 \times S^1)$ into the matrix algebra M_n given by

$$\beta_m(e^{2\pi ikx} e^{2\pi ily}) = T_m^k U_m^l, \qquad \beta_m : C(S^1) \otimes C(S^1) \xrightarrow{as} M_m,$$

where T_m and U_m are the m-dimensional Voiculescu matrices, T_m is a translation and U_m is a diagonal matrix [8]. This asymptotic representation realizes the Bott isomorphism.

The tensor product of $\widetilde{\sigma}$ by β gives an asymptotic representation

$$\widetilde{\sigma}_n \otimes \beta_m : A \otimes C(S^1) \otimes C_0(S^1) \xrightarrow{as} M_{n\times m}.$$

Put $\widehat{\sigma} = P_n \sigma_n P_n$

Theorem 3.1 *There exists a quasi central approximate unity e_n for the exact sequence (3.1) and a sequence of numbers $m(n)$ such that the asymptotic representations ρ_n (3.2) and $\widehat{\sigma}_n \otimes \beta_{m(n)}$ into the matrix algebra $M_{(n+m(n))\times m(n)}$ are equivalent. Moreover for $a \in F \subset A$, $e^{2\pi ix} \in C(S^1)$ and $f \in C_0(S^1)$ the norms*

$$\|\rho_n(a \otimes e^{2\pi ix} \otimes f) - \widehat{\sigma}_n(a) \otimes \beta_{m(n)}(e^{2\pi ix} \otimes f)\|$$

tend to zero.

Proof. Define a sequence of integers $m(n)$ so that the following conditions would be satisfied:

i) $\max_{n\le k\le n+m(n)} \|\sigma_n(a)-\sigma_{n+k}(a)\|$ tend to zero for any $a\in F\subset A$,

ii) $m(n)$ tends to infinity.

As we now assume that $\dim V_n = n$, so $\dim W_{k,l}=1$. We denote the basis vectors of V_n by $v_n^{(j)}$, $j=1,\dots,n$ and the vectors of the subspaces $W_{k,l}$ by $v_{k-l}^{(k)}$. In these denotations the shift operator F constructed in the section 1 acts by the simple formula $Fv_n^{(j)} = v_{n-1}^{(j)}$.

Put

$$a_{n,i}^{(j)} = \begin{cases} 1, & \text{for } \ -n\le i\le n, \\ \frac{m(n)-i+n}{m(n)}, & \text{for } \ n< i\le n+m(n) \ \text{ and } \ j\le n+m(n), \\ 0, & \text{otherwise} \end{cases}$$

and define the diagonal quasi central approximate unity e_n by the formula

$$e_n v_i^{(j)} = a_{n,i}^{(j)} v_i^{(j)}.$$

Notice that e_n exactly commutes with $\sigma_n(a)$ (which are diagonal too) and for any $f\in C_0(S^1)$, $f(0)=0$, $f(e_n)$ almost commutes with F.

Obviously the norms

$$\|\rho_n(a\otimes e^{2\pi ix}\otimes f) - \overline{\rho}_n(a\otimes e^{2\pi ix}\otimes f)\|$$

tend to zero, so we may deal with finite-dimensional asymptotic representations $\overline{\rho}_n$ instead of ρ_n. Direct calculations shows that

$$\overline{\rho}_n(a\otimes e^{2\pi ix}\otimes f) = P_n a' F P_n f(e_n) = P_n(\oplus_{i=n+1}^{m(n)}\sigma_i)(a)P_nFP_nf(e_n) = \oplus_{i=n+1}^{m(n)}\widehat{\sigma}_i(a)FP_nf(e_n),$$

where $a' = \oplus_{i=1}^{\infty}\sigma_i(a)\in\mathcal{E}$ (resp. F) is a lift for a (resp. for $e^{2\pi ix}$), and $f(e_n)$ is the diagonal matrix with elements $\frac{m(n)-i+n}{m(n)}$ on the diagonal. It is easily seen that the norms

$$\|P_nFP_nf(e_n) - T_{m(n)}f(e_n)\|$$

tend to zero and that

$$T_{m(n)}f(e_n) = \beta_{m(n)}(e^{2\pi ix}\otimes f).$$

So it remains to notice that by our choice of the sequence $m(n)$ the norms

$$\|(\oplus_{i=n+1}^{m(n)}\widehat{\sigma}_i)(a) - (\oplus_{i=n+1}^{m(n)}\widehat{\sigma}_n)(a)\|$$

tend to zero too. $\bullet$

Remark. Unfortunately in the case of discrete group C^*-algebras we do not know if it is possible to construct a natural map from the group $\mathcal{R}_a(C^*(\pi)\otimes C(S^1)\otimes C_0(S^1))$ into $K^0(B\pi)$ which would extend the map (1.4) and close the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{R}}_a(\pi) & \longrightarrow & \mathcal{R}_a(C^*[\pi]\otimes C(S^1)\otimes C_0(S^1)) \\ {\scriptstyle\phi}\searrow & & \swarrow{\scriptstyle ?} \\ & K^0(B\pi). & \end{array}$$

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